W2/1 Modules

Donnerstag, 20. Februar 2025 12:06

Remark: (1) $\bigoplus M_i \cong \prod M_i$ if I is finite, A-Mod is an abelian category.

(2) If $(M_i)_{i\in I}$, $M_i \leq M$, then $M' := \sum_{i\in I}' M_i$ is an (internal) direct sum $\Leftrightarrow$ the natural epi $\bigoplus_{i\in I} M_i \to \sum_{i\in I}' M_i$ is an iso.

$\Leftrightarrow$ Every $m \in M'$ has a unique repr $m = \sum_{i\in I'}' m_i$, $\underset{\text{finite}}{I' \subseteq I}$, $m_i \in M_i$

$\Leftrightarrow \forall j:\ M_j \cap \left(\sum_{i\neq j}' M_i\right) = 0.$

Def: M_A is free if $M_A \cong A^{(I)} \cong \bigoplus_{i\in I} A$ for some set I.

A basis of M_A is a family $(m_i)_{i\in I}$ s.t. $A^{(I)} \to M_A$, $(a_i)_{i\in I} \mapsto \sum_{i\in I}' a_i m_i$ is an isomorphism.

Exm: (1) K field $\Rightarrow$ Every K-module is free

(2) $\mathbb{Z}^n_{\mathbb{Z}}$ is free, but: $(\mathbb{Z}/_{n\mathbb{Z}})_{\mathbb{Z}}$ for $n \neq 0$, $\mathbb{Q}_{\mathbb{Z}}$ are not free!

However: $(\mathbb{Z}/_{n\mathbb{Z}})_{\mathbb{Z}/_{n\mathbb{Z}}}$ is free.

Lemma 1.4 (1) Every M_A is a quotient of a free module

[Sketch: $A_A^{(M)} \to M$, $e_m \mapsto m$]

(2) M_A is finitely generated (meaning $M = \langle m_1, \ldots, m_k\rangle_A$ for some $m_i \in M$)

$\Leftrightarrow \exists k \geq 0\ \exists$ epi: $A^k_A \to M_A$

(3) M_A is finitely generated free $\Leftrightarrow \exists k \geq 0:\ M \cong A^k_A$ [there is something small to check for „$\Rightarrow$"]

(easy exercise)

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<u>Lemma 1.5</u> (Nakayama Lemma) Let M_A be a f.g. module

(1) If $\mathfrak{J}(A)M = M$, then $M = 0$.

(2) If $N \leq M$ s.t. $M = N + \mathfrak{J}(A)M$, then $M = N$.

<u>Proof</u>: (1) Assume $M \neq 0$. Let $m_1, \dots, m_r$ be a minimal set of generators for M (then $r \geq 1$). By assumption,

$$m_r = a_1 m_1 + \dots + a_{r-1} m_{r-1} + a_r m_r \quad \text{for some } a_j \in \mathfrak{J}(A)$$

$\Rightarrow (1-a_r) m_r \in \langle m_1, \dots, m_{r-1} \rangle$.

$a_r \in \mathfrak{J}(A) \overset{L1.1}{\Longrightarrow} (1-a_r) \in A^\times \Rightarrow m_r \in \langle m_1, \dots, m_{r-1} \rangle$ ↯ minimality of r.

(2) Apply (1) to M/N, observing $\mathfrak{J}(A)(M/N) = \mathfrak{J}(A)M + N/N = 0$.

$\square$

<u>Def</u>: A is local if $A \neq 0$ and A has a unique maximal ideal $\mathfrak{m}$. <u>Notation</u>: $(A, \mathfrak{m})$

<u>Exm</u>: $\mathbb{Z}_{(2)} = \{ \frac{a}{b} \in \mathbb{Q} : 2 \nmid b \}$ has $\mathfrak{J}(\mathbb{Z}_{(2)}) = 2\mathbb{Z}_{2}$,

and this is the unique max. ideal [if $x = \frac{a}{b} \notin 2\mathbb{Z}_{(2)}$ with $\gcd(a,b) = 1$, then if $2 \nmid a \Rightarrow x^{-1} = \frac{b}{a} \in \mathbb{Z}_{(2)}$, so $x \in \mathbb{Z}_{(2)}^\times$]

If $(A, \mathfrak{m})$ is local:

- $A/\mathfrak{m}$ is a field
- $\mathfrak{m} = \mathfrak{J}(A)$

<u>Cor 1.6</u> Let $(A, \mathfrak{m})$ be local and M_A a f.g. module.

If $x_1, \dots, x_r \in M$ are such that $x_1 + \mathfrak{m}M, \dots, x_r + \mathfrak{m}M$ is a basis of the $A/\mathfrak{m}$-vector space $M/\mathfrak{m}M$, then $x_1, \dots, x_r$ generate

If $x_1,\dots,x_r \in M$ are such that $x_1, \dots, x_r$ mod m is a basis of the A/m-vector space M/mM, then $x_1,\dots,x_r$ generate M.

<u>Exact Sequences</u> A complex $(M_\bullet, f_\bullet)$ is a sequence (finite or infinite) of modules $(M_n)_n$ together with hom's $f_n : M_n \to M_{n+1}$ s.t. $f_{n+1} \circ f_n = 0 \ \forall n$.

$$M_\bullet : \quad \cdots \longrightarrow M_{n-1} \xrightarrow{f_{n-1}} M_n \xrightarrow{f_n} M_{n+1} \longrightarrow \cdots$$

Note: $f_{n+1} \circ f_n = 0 \Leftrightarrow \operatorname{im} f_n \subseteq \ker f_{n+1}$

$M_\bullet$ is exact if $\operatorname{im} f_n = \ker f_{n+1} \ \forall n$.

A short exact sequence (SES) is an exact sequence of the form

$$0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$$

Exactness here means:

- f is injective $(\ker f = \operatorname{im} 0 = 0)$
- $\operatorname{im} f = \ker g$
- g is surjective $(\operatorname{im} g = \ker(0 : P \to 0) = P)$

<u>Exm:</u> ·) $N \leq M$, $\quad 0 \to N \hookrightarrow M \xrightarrow{\pi} M/N \to 0$ is a SES.

$$n \longmapsto n$$
$$m \longmapsto m + N$$

·) $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0 \qquad (n \neq 0)$

$$x \longmapsto nx$$
$$y \longmapsto y + n\mathbb{Z}$$

<u>Remark</u> If $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ is a SES, then $M \cong \ker g$, $P \cong \operatorname{coker} f = N/\operatorname{im} f$. The following diagram commutes and has exact rows:

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$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "f", "\wr"'] & N \arrow[r, "g"] \arrow[d, "id"] & P \arrow[r] \arrow[d, "\wr", "(\hat{g})^{-1}"'] & 0 \\ 0 \arrow[r] & K \arrow[r, hook] & N \arrow[r, "\pi"] & N/K \arrow[r] & 0 \end{tikzcd}$$

with
$K = \operatorname{im} f = \ker g$
$\hat{g}(n+K) = g(n)$.

<u>Lemma 1.7</u> For a SES $0 \to M \xrightarrow{f} N \xrightarrow{g} P \to 0$ TFAE

(a) $\exists$ commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "id"] & N \arrow[r, "g"] \arrow[d, "h"] & P \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & M \arrow[r, hook] & M \oplus P \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

$m \mapsto (m,0)$
$(m,p) \mapsto p$

(b) $\exists$ hom $s: P \to N$ s.t. $g \circ s = id_P$

(c) $\exists$ hom $r: N \to M$ s.t. $r \circ f = id_M$

In this case $h: N \xrightarrow{\sim} M \oplus P$ is an isomorphism. We say the SES splits.

(w/o proof; If you have not seen it yet, it is a good exercise)

<u>Hom-Functors:</u> For $M, N \in A\text{-}\underline{Mod}$, $\operatorname{Hom}(M,N)$ is itself an A-module (pointwise operations).

There are two functors: $\operatorname{Hom}(M,-): A\text{-}\underline{Mod} \to A\text{-}\underline{Mod}$ (covariant),
$\operatorname{Hom}(-,N): (A\text{-}\underline{Mod})^{op} \to A\text{-}\underline{Mod}$ (contravariant)

<u>$\operatorname{Hom}(M,-)$:</u> $A\text{-}\underline{Mod} \ni X \mapsto \operatorname{Hom}(M,X)$

$X \xrightarrow{f} Y \quad \mapsto \quad \operatorname{Hom}(M,X) \xrightarrow{\operatorname{Hom}(M,f) = f_*} \operatorname{Hom}(M,Y)$

with $f_*(g) = f \circ g \quad \forall g \in \operatorname{Hom}(M,X)$

$\underline{\mathrm{Hom}(-,N)}$: $A\text{-}\underline{\mathrm{Mod}} \ni X \longmapsto \mathrm{Hom}(X,N)$

$$X \xrightarrow{f} Y \quad \longmapsto \quad \mathrm{Hom}(Y,N) \xrightarrow{\mathrm{Hom}(f,N) = f^*} \mathrm{Hom}(X,N)$$

with $f^*(g) = g \circ f$

Lemma 1.8 (1) A sequence $0 \to N \xrightarrow{f} M \xrightarrow{g} P$ is exact

$\Leftrightarrow \forall X \in A\text{-Mod}: \; 0 \to \mathrm{Hom}(X,N) \xrightarrow{f_*} \mathrm{Hom}(X,M) \xrightarrow{g_*} \mathrm{Hom}(X,P)$ is exact

(2) A sequence $N \xrightarrow{f} M \xrightarrow{g} P \to 0$ is exact

$\Leftrightarrow \forall X \in A\text{-Mod}: \; 0 \to \mathrm{Hom}(P,X) \xrightarrow{g^*} \mathrm{Hom}(M,X) \xrightarrow{f^*} \mathrm{Hom}(N,X)$ is exact

Proof Easy to check; E.g.

$\underline{(2), \text{"}\Leftarrow\text{"}}$: (i) $0 = f^* \circ g^* = (g \circ f)^* \Rightarrow g \circ f \circ \varphi = 0 \quad \forall \varphi: N \to X$

Using $\varphi = \mathrm{id}_N: N \to N$, we get $g \circ f(n) = 0 \;\; \forall n \in N \Rightarrow \underline{g \circ f = 0}$

(ii) Let $\pi: M \to M/\mathrm{im}\, f \Rightarrow f^*(\pi) = \pi \circ f = 0 \Rightarrow \pi = g^*(\varphi) = \varphi \circ g$ for

some $\varphi: P \to M/\mathrm{im}\, f$. Therefore $\underline{\ker(g)} \subseteq \ker(\pi) = \underline{\mathrm{im}\, f}$

(iii) Let $\pi: P \to P/\mathrm{im}\, g \Rightarrow 0 = \pi \circ g = g^*(\pi) \xRightarrow{g^* \text{ mono}} \pi = 0 \Rightarrow P = \mathrm{im}\, g$. $\square$

The directions "$\Rightarrow$" in L 1.8 mean that $\mathrm{Hom}(X,-)$ and $\mathrm{Hom}(-,X)$ are left exact functors.

Tensor Products Let $M_1, \dots, M_n, P$ be A-modules. A map $f: M_1 \times \dots \times M_n \to P$ is A-multilinear (A-bilinear if $n=2$) if

$$\forall 1 \le i \le n: \quad f(m_1, \dots, m_{i-1}, \underline{a m_i + b m_i'}, m_{i+1}, \dots, m_n)$$
$$= a f(m_1, \dots, m_{i-1}, \underline{m_i}, m_{i+1}, \dots, m_n) + b f(m_1, \dots, \underline{m_i'}, \dots, m_n)$$

$(\forall m_j \in M_j, \; m_i' \in M_i, \; a, b \in A)$

The tensor product $M_1 \otimes \dots \otimes M_n$ is the A-module together

The tensor product $M_1 \otimes \dots \otimes M_n$ is the A-module together with the multilinear map $\otimes : M_1 \times \dots \times M_n \to M_1 \otimes \dots \otimes M_n$, $(m_1, \dots, m_n) \mapsto m_1 \otimes \dots \otimes m_n$ defined by the following UP:

For every A-module P and every multilinear $f: M_1 \times \dots \times M_n \to P$, there exists a unique A-hom $\hat{f}: M_1 \otimes \dots \otimes M_n \to P$ s.t. $\hat{f} \circ \otimes = f$

$$\begin{tikzcd}
M_1 \times \dots \times M_n \arrow[r, "\otimes"] \arrow[dr, "f"'] & M_1 \otimes \dots \otimes M_n \arrow[d, dashed, "\exists! \hat{f}"] \\
 & P
\end{tikzcd}$$

multilinear

hom (= linear)

$$M \otimes_A N = \left\{ \sum_{i=1}^{s} m_i \otimes n_i \ : \ m_i \in M, n_i \in N \right\}$$

⚠ Sums are really necessary

Exm: ⚠ In $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$: $\quad 2 \otimes \bar{1} = 1 \otimes 2 \cdot \bar{1} = 1 \otimes \bar{0} = 0$

In $2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$: $\quad 2 \otimes \bar{1} \neq 0 \qquad [2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},\ (2x, \bar{y}) \mapsto x\bar{y}$ is bilinear$]$

Associativity of $\otimes$: $M_1 \otimes (M_2 \otimes M_3) \cong (M_1 \otimes M_2) \otimes M_3 \cong M_1 \otimes M_2 \otimes M_3$,

Commutativity of $\otimes$: $M_1 \otimes M_2 \cong M_2 \otimes M_1 \qquad$ (naturally)

Functorial in each component: For $M \in A\text{-Mod}$,

$M \otimes - : A\text{-Mod} \to A\text{-Mod}$ is a functor with

$$X \longmapsto M \otimes X,$$

$$X \xrightarrow{f} Y \longmapsto \mathrm{id} \otimes f : M \otimes X \to M \otimes Y,\ m \otimes x \mapsto m \otimes f(x)$$

Thm 1.9 (Hom-$\otimes$ Adjunction) Let $M \in A\text{-Mod}$. Then $- \otimes M$ is left adjoint to $\mathrm{Hom}(M, -)$. Explicitly, $\forall M, N, P$, there are A-isomorphisms

$$\mathrm{Hom}(N \otimes M, P) \xrightarrow{\sim} \mathrm{Hom}(N, \mathrm{Hom}(M, P))$$

$$f \longmapsto n \mapsto (m \mapsto f(n \otimes m))$$

$$n \otimes m \mapsto g(n)(m) \quad \longleftarrow\!\!\shortmid \quad g$$

and these isos are natural transformations in N, M, P.

<u>Proof Strategy:</u> Use UP to construct the $\leftrightarrow$ maps. Then check all the properties are satisfied (straightforward but tedious).

<u>Cor 1.10</u> (1) $M \otimes -$ is right exact, i.e. for every exact sequence $N \to P \to Q \to 0$, also $M \otimes N \to M \otimes P \to M \otimes Q \to 0$ is exact.

(2) $M \otimes \left(\bigoplus_{i \in I} N_i \right) \cong \bigoplus_{i \in I} (M \otimes N_i)$ for all families $(N_i)_i$

<u>Proof:</u> (1) We have $\mathrm{Hom}(M \otimes X, Y) \simeq \mathrm{Hom}(M, \mathrm{Hom}(X, Y)) \ \forall X, Y$.

$N \to P \to Q \to 0$ exact

$\overset{L1.8}{\Rightarrow}$ $0 \to \mathrm{Hom}(Q, X) \to \mathrm{Hom}(P, X) \to \mathrm{Hom}(N, X)$ exact.

$$\overset{L1.8}{\Rightarrow} \begin{array}{ccccccc} 0 \to & \mathrm{Hom}(M, \mathrm{Hom}(Q,X)) & \to & \mathrm{Hom}(M, \mathrm{Hom}(P,X)) & \to & \mathrm{Hom}(M, \mathrm{Hom}(N,X)) & \text{exact} \\ & \wr\Vert & & \wr\Vert & & \wr\Vert & \\ 0 \to & \mathrm{Hom}(M \otimes Q, X) & \longrightarrow & \mathrm{Hom}(M \otimes P, X) & \longrightarrow & \mathrm{Hom}(M \otimes N, X) & \end{array}$$

$\overset{L1.8}{\Rightarrow}$ $M \otimes N \to M \otimes P \to M \otimes Q \to 0$ exact.

(2) Construct the isomorphism using UP of $\oplus$ and of $\otimes$ (exercise)

<u>Or:</u> Use adjoints again on $\mathrm{Hom}(-, X)$

(left adjoints commute with colimits). $\square$